

Cauchy Transforms and Multipliers

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In this paper we prove a number of results on Cauchy transforms and multipliers. For example, we prove that if $f = IF \in \mathcal{F}_0$, where I is an inner function satisfying a certain condition and F is outer, then $F \in \mathcal{F}_0$. © 1997 Academic Press

1. INTRODUCTION

Let U be the open unit disk and let $T = \{z: |z| = 1\}$. Let \mathcal{M} denote the set of complex valued Borel measures on T . For each $\alpha > 0$ let \mathcal{F}_α denote the family of functions f having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = \int_T \frac{1}{(1 - \bar{\zeta}z)^\alpha} d\mu(\zeta), \quad (1)$$

for $|z| < 1$. For $\alpha = 0$, let \mathcal{F}_0 denote the family of functions f having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = \int_T \log \frac{1}{1 - \bar{\zeta}z} d\mu(\zeta) + f(0). \quad (2)$$

\mathcal{F}_α is a Banach space with respect to the norm

$$\begin{cases} \|f\|_{\mathcal{F}_\alpha} = \inf \|\mu\|, & \text{for } \alpha > 0 \\ \|f\|_{\mathcal{F}_0} = \inf \|\mu\| + |f(0)|, \end{cases} \quad (3)$$

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where μ varies over all measures in \mathcal{M} for which (1) or (2) hold and where $\|\mu\|$ denotes the total variation norm of μ [3, 6].

A function f analytic in U is called a multiplier of \mathcal{F}_α provided that $fg \in \mathcal{F}_\alpha$ for all $g \in \mathcal{F}_\alpha$. Let M_α denote the set of all multipliers of \mathcal{F}_α . Let

$$\|f\|_{M_\alpha} = \sup\{\|fg\|_{\mathcal{F}_\alpha} : g \in \mathcal{F}_\alpha, \|g\|_{\mathcal{F}_\alpha} \leq 1\}. \quad (4)$$

Note that

$$\|fg\|_{M_\alpha} \leq \|f\|_{M_\alpha} \cdot \|g\|_{M_\alpha}. \quad (5)$$

Hence M_α with this norm is a Banach algebra [2, p. 182; 7; 8; 6].

In Section 2 we show for w analytic in U and $f \in M_\alpha$, with $\|f\|_{M_\alpha} < 1$ that $w \circ f \in M_\alpha$. This implies that $f \in H^\infty$, for all $f \in M_\alpha$ and provides many examples of functions in M_α .

In [6] it was proved that if $f' \in H^1$ then $f \in M_\alpha$ for all $\alpha > 0$. In [4] it was proved that if $f' \in H^p$ for some $p > 1$ then $f \in M_0$. Both statements are sufficient but not necessary. In [5] an example was given of a function $f \in M_0$ which is not continuous on \bar{U} . In Section 3, we describe the boundary behavior of $f \in M_\alpha$. To be specific, we show that at each point of the boundary of U , the limit exists from the inside of each orocycle at the same point.

It is known [1, 2] that if $f = IF \in H^p$, where I is inner and F is outer, then $\|f\|_p = \|F\|_p$. In [8] it was shown that if $f = IF \in \mathcal{F}_1$, where I is inner and F is outer, then $\|F\|_{\mathcal{F}_1} \leq \|f\|_{\mathcal{F}_1}$. Recently, D. Hallenbeck and K. Samotij conjectured (private communication) that if $f = IF \in \mathcal{F}_\alpha$, where $0 \leq \alpha \leq 1$, I is inner, and F is outer then $F \in \mathcal{F}_\alpha$.

In Section 4, we prove that if $f = IF \in \mathcal{F}_0$ and I satisfies condition (12) then $F \in \mathcal{F}_0$.

2. M_α AS A BANACH ALGEBRA

We can easily show that M_α , $\alpha \geq 0$, is a complex Banach algebra with norm as in (4) and (5), see [2, p. 182]. Let M_α^* denote the dual space of M_α and \mathcal{A}_α denote the subset of the unit ball of M_α^* consisting of elements that are also multiplicative, see [2, p. 184]. Define the Gelfand transform of $f \in M_\alpha$, by

$$\hat{f}(m) = m(f),$$

for all $m \in \mathcal{A}_\alpha$ and

$$\|\hat{f}\| = \sup_{m \in \mathcal{A}_\alpha} |\hat{f}(m)|.$$

As

$$\|f\|_{M_\alpha} = \sup_{L \in M_\alpha^*, \|L\|=1} |L(f)|,$$

$\|\hat{f}\| \leq \|f\|_{M_\alpha}$. We say that \hat{f} is an isometry when $\|\hat{f}\| = \|f\|_{M_\alpha}$. This is equivalent to the condition $\|f^2\|_{M_\alpha} = \|f\|_{M_\alpha}^2$ [2, p. 185]. Unfortunately not every \hat{f} is an isometry, because it was shown that

$$n^{1-\alpha} \leq \|z^n\|_{M_\alpha} \leq cn^{1-\alpha},$$

for all $0 \leq \alpha \leq 1$, see [3]. Certainly, $\|z\|_{M_\alpha} > 1$. Hence $\|z^n\|_{M_\alpha} < \|z\|_{M_\alpha}^n$, for all $n \geq 1$ and consequently, \hat{z} is not an isometry. We conjecture that the constant functions are the only isometries in M_α .

Now we recall the following general theorem [2, p. 182]

THEOREM A. *Let A be a Banach algebra with norm $\|\cdot\|$ and let m be a complex homomorphism of A . Then m is a continuous linear functional with norm*

$$\|m\| = \sup_{\|f\| \leq 1} |m(f)| \leq 1.$$

Remark. Theorem A implies that every complex homomorphism of M_α is an element of \mathcal{A}_α .

This theorem immediately gives another proof for the following known result [3, 7].

THEOREM B. $|f(z)| \leq \|f\|_{M_\alpha}$, for all $z \in U$, $f \in M_\alpha$, and where $\alpha \geq 0$.

Proof. Fix $z \in U$ and let $m(f) = f(z)$. Then m is a complex homomorphism on M_α . This and Theorem A give the result.

The following theorems concern subordination:

THEOREM 1. *Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in U and $f \in M_\alpha$, where $\alpha \geq 0$. If $\|f\|_{M_\alpha} < 1$ then $\phi \circ f \in M_\alpha$ and $\sum_{n=0}^{\infty} a_n f^n$ converges to $\phi \circ f$ in M_α .*

Proof. Let ϕ_k denote the k th partial sum of the Taylor series of ϕ . Use (5) to get

$$\|f^n\|_{M_\alpha} \leq \|f\|_{M_\alpha}^n.$$

Hence

$$\|\phi_k \circ f\|_{M_\alpha} \leq \sum_{n=0}^{\infty} |a_n| \|f\|_{M_\alpha}^n < \infty,$$

for all k . This is because $\|f\|_{M_\alpha} < 1$ and ϕ analytic. Since $\|\phi_k \circ f\|_{M_\alpha}$ is uniformly bounded and $\phi_k \circ f \rightarrow \phi \circ f$ locally uniformly we conclude that $\phi \circ f \in M_\alpha$ and the series converges to $\phi \circ f$ in M_α , see [3].

THEOREM 2. *Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in U , with $\sum_0^{\infty} |a_n| < \infty$. Let $f \in M_\alpha$, where $\alpha \geq 0$. Then each of the following follows*

(a) *If $\|f\|_{M_\alpha} \leq 1$ then $\phi \circ f \in M_\alpha$ and $\sum_{n=0}^{\infty} a_n f^n$ converges to $\phi \circ f$ in M_α .*

(b) *If $\|f^n\|_{M_\alpha}$ is uniformly bounded for all $n \geq 1$ then $\phi \circ f \in M_\alpha$.*

Proof. The proof is similar to the proof of Theorem 1.

It is not easy to deal with the M_α -norm. To be specific, we could not express $\|\phi \circ f\|_{M_\alpha}$ in terms of $\|f\|_{M_\alpha}$. However, if we use the Gelfand-norm then we have:

THEOREM 3. *Let ϕ be as in the statement of Theorem 2 and let $f \in M_\alpha$, with $\|f\|_{M_\alpha} \leq 1$. Then*

$$\|\phi \circ \hat{f}\| = \|\phi(\|\hat{f}\|z)\|_\infty.$$

Proof. Let $m \in \mathcal{A}_\alpha$. Then by theorem 2, as $\sum_0^{\infty} |a_n| < \infty$, $\phi \circ f \in M_\alpha$ and so

$$m(\phi \circ f) = \sum_0^{\infty} a_n (m(f))^n. \quad (6)$$

But, as $\|\hat{f}(m)\| \leq 1$,

$$\{e^{i\theta} m(f) : m \in \mathcal{A}_\alpha\} = \{z : |z| \leq \|\hat{f}\|\}.$$

This and (6) imply the result.

3. BOUNDARY BEHAVIOR OF $f \in M_\alpha$

It was shown in [6] that if $f' \in H^1$ the $f \in M_\alpha$ for all $\alpha > 0$. In [4] it was proved that $f' \in H^p$ for some $p > 1$ implies $f \in M_0$ while $f' \in H^1$ does not imply $f \in M_0$. In [5] an example was constructed of a function in M_0 that is not continuous on \bar{U} . In the following theorem we show that functions in M_0 are almost continuous on \bar{U} . Let $D_t(\theta)$, $1 > t > 0$, be the

region inside the orocycle $C_t(\theta)$ given by

$$C_t(\theta) = \left\{ z: \frac{1 - |z|^2}{|e^{i\theta} - z|^2} = t \right\}$$

and inside the circle $\{z: |z - e^{i\theta}| < 1/2\}$. Then we have the following theorem

THEOREM 4. *Let $f \in M_0$. Then*

$$\lim_{z \rightarrow e^{i\theta}} f(z) \text{ exists, } \quad z \in \overline{D_t(\theta)}.$$

Proof. Choose $e^{i\theta} = 1$ and assume that $f(0) = 0$. Then

$$f(z) \log \frac{1}{1-z} \in \mathcal{F}_0.$$

Hence

$$f(z) \log \frac{1}{1-z} = \int_T \log \frac{1}{1-\bar{x}z} d\mu(x)$$

and

$$f(z) = \int_T \frac{\log(1-\bar{x}z)}{\log(1-z)} d\mu(x). \quad (7)$$

Choose $z \in \overline{D_t(1)}$. Then

$$\frac{1 - |z|^2}{|1 - z|^2} \geq t$$

and $|1 - z| < 1/2$. Take logarithms to conclude that

$$\frac{\log(1 - |z|^2)}{\log|1 - z|} \leq \frac{\log t}{\log|1 - z|} + 2. \quad (8)$$

If we assume that the branches of all logarithms in (7) are principal then the arguments are all bounded by $\pi/2$ and hence

$$\begin{aligned} \left| \frac{\log(1 - \bar{x}z)}{\log(1 - z)} \right| &\leq \frac{\log(1 - |z|)}{\log|1 - z|} + \frac{\log 2 + \pi/2}{|\log|1 - z||} \\ &\leq \frac{|\log t| + \log 2 + \pi/2}{|\log|1 - z||} + 2. \end{aligned}$$

The last inequality follows from (8). This implies that the integrand in (7), as $z \rightarrow 1$, converges to the function

$$F(x) = \begin{cases} 0, & \text{for } x \neq 1 \\ 1, & \text{for } x = 1. \end{cases}$$

Hence we conclude, using the Lebesgue dominated convergence theorem, that

$$\lim_{z \rightarrow 1} f(z) = \mu(\{1\}).$$

Remark. The limit of Theorem 4 remains the same when we replace the orocycle by a path that is more tangential. Let $(1+z)/(1-z) = u + iv$ and let

$$L_t = \{z: uv = t, |1-z| < 1/2\}.$$

Then $L_t \rightarrow 1$ as $u \rightarrow 0$.

Assume that $t > 0$ and let $z \in L_t$. Then

$$\frac{u^4 + t^2}{u^2} = u^2 + \frac{t^2}{u^2} = \left| \frac{1+z}{1-z} \right|^2. \quad (9)$$

Take logarithms in (9), to get

$$\log|u^4 + t^2| = 2 \log u + 2 \log \left| \frac{1+z}{1-z} \right|. \quad (10)$$

Assume that $u < 1$. Then $\log t^2 < \log|u^4 + t^2| < \log(t^2 + 1)$. Write $\log u = \log((1-|z|^2)/|1-z|^2)$. Then (10) gives

$$\frac{\log(1-r^2)}{\log|1-z|} \leq c,$$

for some constant c . This is similar to (8). Continue as in the proof of Theorem 4 to conclude that

$$\lim_{z \rightarrow 1} f(z) = \mu(\{1\}),$$

along L_t .

4. OUTER PART AND INNER PART OF $f \in \mathcal{F}_0$

It was shown in [8] that if $f \in \mathcal{F}_1$ and

$$f = IF, \quad (11)$$

where F is outer and I is inner then $\|F\|_{\mathcal{F}_1} \leq \|f\|_{\mathcal{F}_1}$.

In this section we prove:

THEOREM 5. *If $f \in \mathcal{F}_0$ is as in (11) and I satisfies*

$$\int_T \int_0^1 |I'(tx)| dt d\mu(x) < C \|zf\|_{\mathcal{F}_0}, \quad (12)$$

for all μ that give zf as in (2), then $\|F\|_{\mathcal{F}_0} \leq C_1 \|zf\|_{\mathcal{F}_0}$, where C, C_1 are uniform constants.

This is a consequence of the following more general proposition.

PROPOSITION 1. *If $f = IF \in \mathcal{F}_0$, where I is an inner function satisfying (12) then $\|F\|_{\mathcal{F}_0} \leq C_1 \|zf\|_{\mathcal{F}_0}$.*

Proof. The Frostman Theorem [2] implies that there is a sequence of Blaschke products converging, in the H^∞ -norm, to I of the form

$$I_\zeta = \frac{I - \zeta}{1 - \bar{\zeta}I}.$$

To be specific, $\|I_\zeta - I\|_\infty < \epsilon$ when $|\zeta| < \delta$. We may choose a sequence of ζ converging to 0 so that each I_ζ has only simple zeros. Put

$$I_\zeta = B(z) = \prod_{j=1}^{\infty} \frac{z - a_j}{1 - \bar{a}_j z}$$

and let B_n be the subproduct from $j = 1$ to $j = n$. Since it is known [4] that $f \in \mathcal{F}_0$ is equivalent to $zf \in \mathcal{F}_0$, we put

$$zf(z) = \int_T \int_0^1 \frac{\bar{x}z}{1 - t\bar{x}z} dt d\mu(x), \quad (13)$$

where $0 \leq t < 1$ and $\mu \in M$ given zf as in (2). Then

$$f(\rho z) = \int_T \int_0^\rho \frac{\bar{x}}{1 - t\bar{x}z} dt d\mu(x), \quad (14)$$

where $0 \leq t \leq \rho < 1$. Put

$$\frac{\bar{x}}{(1 - t\bar{x}z)} \frac{1}{B_n(z)} = \frac{\bar{x}/B_n(1/t\bar{x})}{1 - t\bar{x}z} + \sum_{j=1}^n \frac{(1 - |a_j|^2)\bar{x}}{B_j(a_j)(z - a_j)(1 - \bar{t}\bar{x}a_j)}, \quad (15)$$

where $B_j(a_j) = \prod_{1 \leq k \neq j \leq n} ((a_k - a_j)/(1 - \bar{a}_k a_j))$. Note that

$$\frac{1}{B_n(1/t\bar{x})} = \overline{B_n(tx)}$$

and, from (14), that

$$\int_T \int_0^\rho \frac{\bar{x}}{1 - t\bar{x}a_j} dt d\mu(x) = I(\rho a_j) F(\rho a_j),$$

for all j . Integrate (15) with respect to dt , $d\mu$ and then take limits to get

$$\begin{aligned} \frac{f(\rho z)}{I(z)} &= \lim_{\zeta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{f(\rho z)}{B_n(z)} \\ &= \lim_{\zeta \rightarrow 0} \lim_{n \rightarrow \infty} \int_T \int_0^\rho \frac{\bar{x} \overline{B_n(tx)}}{1 - t\bar{x}z} dt d\mu(x) \\ &\quad + \lim_{\zeta \rightarrow 0} \lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{(1 - |a_j|^2) I(\rho a_j) F(\rho a_j)}{B_j(a_j)(z - a_j)} \\ &= A_1 + A_2. \end{aligned} \tag{16}$$

We first consider A_1 . As $\rho < 1$, (16) implies that

$$A_1 = \int_T \int_0^\rho \frac{\bar{x} \overline{I(tx)}}{1 - t\bar{x}z} dt d\mu(x). \tag{17}$$

Integrate (17), using integration by parts with respect to dt , to get

$$\begin{aligned} zA_1 &= \int_T \log \frac{1}{1 - \bar{x}\rho z} (\overline{I(\rho x)} d\mu(x)) \\ &\quad - \int_0^\rho \int_T \bar{x} \log \frac{1}{1 - t\bar{x}z} \overline{I'(tx)} d\mu(x) dt. \end{aligned} \tag{18}$$

For each $t < 1$,

$$\frac{1}{1 - t\bar{x}z} = \int_0^{2\pi} \frac{1}{1 - \bar{y}z} \Re \frac{1 + txe^{-i\theta}}{1 - txe^{-i\theta}} d\theta / 2\pi, \tag{19}$$

for $z \in U$ and $y = e^{i\theta}$. Integrate (19) with respect to z to get

$$\log \frac{1}{1 - t\bar{x}z} = \int_0^{2\pi} \frac{t\bar{x}}{\bar{y}} \log \frac{1}{1 - \bar{y}z} \Re \frac{1 + txe^{-i\theta}}{1 - txe^{-i\theta}} d\theta / 2\pi. \tag{20}$$

Let

$$\Phi_\rho(\theta) = \int_T \int_0^\rho \frac{t\bar{x}^2}{\bar{y}} \Re \frac{1 + txe^{-i\theta}}{1 - txe^{-i\theta}} \overline{I'(tx)} dt d\mu(x). \tag{21}$$

Then

$$\begin{aligned} & \int_0^\rho \int_T \bar{x} \log \frac{1}{1 - t\bar{x}z} \overline{I'(tx)} d\mu(x) dt \\ &= \int_0^{2\pi} \log \frac{1}{1 - \bar{y}z} \Phi_\rho(\theta) d\theta/2\pi. \end{aligned} \quad (22)$$

Now we show that the L^1 -norm of Φ_ρ is uniformly bounded. Note that

$$\int_0^{2\pi} \Re \frac{1 + txe^{-i\theta}}{1 - txe^{-i\theta}} e^{i\theta} d\theta/2\pi = tx.$$

Hence it follows from (21), by using (12), that

$$\int_0^{2\pi} |\Phi_\rho(\theta)| d\theta/2\pi \leq \int_T \int_0^\rho |\overline{I'(tx)}| dt |d\mu(x)| \leq C \|zf\|_{\bar{\rho}_0}.$$

This implies, by using the Helly selection theorem, that there is a subsequence of $\rho \rightarrow 1$ so that $\Phi_\rho d\theta$ converges weak star to some $\nu_1 \in \mathcal{M}$.

Now replace t by ρ in (20) and let

$$\Psi_\rho(\theta) = \int_T \frac{\rho \bar{x}^2}{\bar{y}} \Re \frac{1 + \rho xe^{-i\theta}}{1 - \rho xe^{-i\theta}} \overline{I(\rho x)} d\mu(x). \quad (23)$$

Clearly, the L_1 -norm of Ψ_ρ is uniformly bounded and so Ψ_ρ converges weak star to some measure $\nu_2 \in \mathcal{M}$. Hence

$$\int_T \log \frac{1}{1 - \bar{x}\rho z} (\overline{I(\rho x)} d\mu(x)) = \int_0^{2\pi} \log \frac{1}{1 - \bar{y}z} \Psi_\rho(\theta) d\theta/2\pi \quad (24)$$

and then (18) becomes, by using (22) and (24)

$$zA_1 = \int_T \log \frac{1}{1 - \bar{x}z} \Psi_\rho(x) d\theta - \int_T \log \frac{1}{1 - \bar{x}z} \Phi_\rho(\theta) d\theta.$$

Hence

$$\lim_{\rho \rightarrow 1} zA_1 = \int_T \log \frac{1}{1 - \bar{x}z} d(\nu_1(x) - \nu_2(x)). \quad (25)$$

Second, we find A_2 . It is not hard to notice that, see [1],

$$\frac{1}{2\pi} \int_T \frac{f(\rho w) \overline{B_n(w)}}{w - z} dw = \sum_{j=1}^n \frac{(1 - |a_j|^2) f(\rho a_j)}{B_j(a_j)(a_j - z)} + \frac{f(\rho z)}{B_n(z)}. \quad (26)$$

Since $|f(\rho w)\overline{B_n(w)}| \leq |f(\rho w)|$ and since $f \in H^1$, we conclude using the Lebesgue dominated convergence theorem that the limit, as $n \rightarrow \infty$, $\zeta \rightarrow 0$, and $\rho \rightarrow 1$, of the left side of (26) is equal to $F(z)$. Hence, from (26) and (16)

$$\lim_{\rho \rightarrow 1} A_2 = F(z) - F(z) = 0. \quad (27)$$

Therefore (27), (26), and (16) imply that there is a measure $\nu \in \mathcal{M}$ so that

$$F(z) = \int_T \log \frac{1}{1 - \bar{x}z} d\nu(x).$$

Hence $F \in \mathcal{F}_0$.

Remark. (1) The condition (12) is satisfied by any inner function in M_α for some $\alpha \geq 0$. This is because for any $f \in M_\alpha$

$$\int_0^1 |f'(re^{i\theta})| dr \leq \|f\|_{M_\alpha},$$

for all θ [6, 8].

(2) We believe that condition (12) in Proposition 1 can be removed.

Finally we state the following corollary

COROLLARY. *If $f = IF \in M_0$, where I is an inner function satisfying the condition*

$$\int_T \int_0^1 |I'(tx)| dt d\mu(x) \leq C \|zf\|_{M_0},$$

for all μ that give zf as in (2) then

$$\|F\|_{M_0} \leq C_1 \|zf\|_{M_0}.$$

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